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Ionization equilibrium and EOS of a low-temperature hydrogen plasma in weak magnetic fields

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Abstract. The influence of a constant uniform magnetic field on the ionization equilibrium and the thermodynamic properties of a nondegenerate partially ionized hydrogen plasma is studied for weak magnetic fields. Using the methods developed in a previous work, various interaction contributions to the thermodynamic functions are given. The equation of state of a quantum magnetized plasma is presented within the framework of a virial expansion up to the second order in the fugacities, including ladder type contributions corresponding to bound states. A simple interpolation formula for an effective partition function is proposed, connecting the low- and high-field results. Furthermore, a closed analytical approximation for the thermodynamic functions in the chemical picture and a Saha equation for weakly magnetized plasmas are presented.

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1 Introduction

The calculation of the equation of state (EOS) of a multi-component quantum plasma consisting of charged particles interacting *via* the Coulomb potential is of theoretical interest as well as of practical relevance, *e.g.* for astrophysical systems such as stars. The aim of this paper is the discussion of the ionization equilibrium in the temperature region of 10^3-10^5 K. We employ here two methods:

(i) implicit representation of the pressure in the grand canonical ensemble by using a fugacity expansion [1–3],
(ii) the method of nonideal Saha equation [3,4].

The main difference from earlier investigations is that we study in this work the influence of weak magnetic fields on the ionization equilibrium of a hydrogen plasma by using recently obtained analytical results for the virial coefficients [5,6]. The ionization equilibrium in strong magnetic fields was discussed by Lai and Salpeter [7], Khersonskii [8], Miller [9] and Potekhin *et al.* [10]. The present treatment includes nonideality effects and is based on earlier results concerning a low-density, low-fugacity expansion for the equation of state (EOS) of a two-component plasma embedded in an external constant magnetic field [5,6]. Related problems were also treated by Cornu [11] and Boose and Perez [12] who derived a virial expansion of the EOS by using a formalism, which is based on the Feynman-Kac path integral representation of the grand canonical potential. In our previous work, we employed the method of Green's function [5] and virial expansion in the grand canonical ensemble [6]. Here, we will use these results and focus our attention on the region where the plasma is partially ionized and nondegenerated. First, we will use an expansion of the magnetized plasma pressure in terms of the fugacity $z = \exp(\beta\mu)$ in order to obtain the EOS of a weakly coupled magnetized plasma. Thus, we can derive explicit expressions for various contributions to the quantum second virial coefficient. Although the formalism is formally valid only for low densities, the obtained explicit expressions are appropriate even at sufficient high densities since the magnetic field increases the domain of classical behavior towards higher densities. The second virial coefficient contains both scattering and bound state contributions of two-particle states. Being interested in the thermodynamic properties of quantum magnetized plasmas the influence of the magnetic field on the energy eigenstates of a two-particle system has to be taken into account.

Usually the magnetic field is measured by the dimensionless parameter $\gamma = \hbar\omega_c/(2Ry) = B/B_0$, where $\hbar\omega_c$ is the electron cyclotron energy, $B_0 = 2.35 \times 10^5$ T, and $Ry = e^2/(8\pi\epsilon_0 a_B) \approx 13.605$ eV is the ionization energy of the field-free hydrogen atom. Whenever $\gamma > 1$, *i.e.* the cyclotron energy is larger than the typical Coulomb energy, the structure of the hydrogen atom is dramatically changed. This problem has been approached by several authors [7,13,14]. Using their results for the asymptotic behavior of the ground state energy, we will propose an interpolation formula for the effective partition function in order to study the thermodynamic properties of magnetized plasmas at arbitrary field strengths.

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Recently, the problem of ionization equilibrium of hydrogen atoms in superstrong magnetic fields ($\gamma \gg 1$) was considered by Lai and Salpeter [7]. They proposed an ideal Saha equation of a hydrogen gas including bound states but neglecting screening effects and scattering contributions to the second virial coefficient. Using the EOS obtained in our derivation, we construct a modified Saha equation which takes into account nonideality effects as well.

The paper is organized as follows. In Section 2, we discuss the representation of thermodynamic functions by a fugacity expansion. In Section 3, we discuss in more detail the bound state contributions in the fugacity representation. In Section 4, we consider the transition to a chemical description. Based upon this, we obtain in Section 5 a Saha equation and discuss the ionization equilibrium.

2 Representation of the thermodynamic functions by a fugacity expansion

We consider a two-component charge-symmetrical system of N spin half particles of charge $(-e)$ and mass m_e and N spin half particles of charge e and mass m_i. The Hamilton operator of our systems consists of two particle contributions. Each pair of species a and b contributes

$$\widehat{H}_{ab}^{\lambda} = \left(\frac{(\mathbf{p}_a - e_a\mathbf{A}_a)^2}{2m_a} + \mu_{\mathrm{B}}^a B_0\sigma_z\right) + \left(\frac{(\mathbf{p}_b - e_b\mathbf{A}_b)^2}{2m_b} + \mu_{\mathrm{B}}^b B_0\sigma_z\right) + \lambda V_{ab}(\mathbf{r}), \quad \sigma_z = -1, +1 \tag{1}$$

with the Coulombic interaction potential

$$V_{ab}(\mathbf{r}) = \frac{e_a e_b}{4\pi\epsilon_0 r}\,. \tag{2}$$

The pressure can be split into ideal contributions and interaction contributions

$$p = p_{\mathrm{id}} + p_{\mathrm{int}}. \tag{3}$$

In the case without Coulombic interaction $e^2 = 0$, the pressure and the particle density of the plasma in a homogeneous magnetic field $\mathbf{B} = (0, 0, B_0)$ are given by a sum of Fermi integrals over all Landau levels n

$$p_{\mathrm{id}} = kT\sum_a \frac{4x_a}{\Lambda_a^3}\frac{1}{\sqrt{\pi}}{\sum_{N=0}}' F_{1/2}(\ln z_N^a),$$
$$n = \sum_a \frac{2x_a}{\Lambda_a^3}\frac{1}{\sqrt{\pi}}{\sum_{N=0}}' F_{-1/2}(\ln z_N^a) \tag{4}$$

($x_a = \hbar\omega_{\mathrm{c}}^a/(2kT)$ with $\omega_{\mathrm{c}}^a = |e_a|B_0/m_a$, $\Lambda_a = h/\sqrt{2\pi m_a kT}$, and $z_N^a = \exp\left[\beta(\mu - N\hbar\omega_{\mathrm{c}}^a)\right]$). The prime indicates the double summation due to the spin degeneracy except for the $N = 0$ level.

The interaction part of the pressure will be expressed in terms of a fugacity expansion which we cut after the second virial coefficient [1,3]

$$\beta p_{\mathrm{int}} = \frac{\kappa^3}{12\pi} + \sum_{ab}\tilde{z}_a\tilde{z}_b\left(\frac{\pi}{3}\lambda_{ab}^3\xi_{ab}^3\ln(\kappa\lambda_{ab}) + \frac{\pi}{2}\beta^3\frac{e_a^2}{4\pi\epsilon_0}\frac{e_b^4}{4\pi\epsilon_0} + B_{ab}\right) + O(\tilde{z}^{5/2}\ln\tilde{z}). \tag{5}$$

Here, we have introduced the interaction parameter $\xi_{ab} = -e_a e_b/4\pi\epsilon_0 kT\lambda_{ab}$ (with $\lambda_{ab} = \hbar/\sqrt{2m_{ab}kT}$ and $m_{ab} = m_a m_b/(m_a + m_b)$) and the modified fugacities

$$\tilde{z}_a = z_a\,\frac{2}{\Lambda_a^3}\,\frac{x_a}{\tanh(x_a)}\,. \tag{6}$$

In the limit of small densities, we have $\tilde{z}_a \to n_a$. the first term on the right-hand side (RHS) of equation (5) is the Debye contribution in the grand canonical ensemble. The squared inverse of the Debye radius is given by $\kappa^2 = \beta(e^2/\epsilon_0)(\tilde{z}_e + \tilde{z}_i)$. We focus on the calculation of the second virial coefficient B_{ab}. In the following it will be convenient to divide the second virial coefficient into the direct part B_{ab}^{di} and the exchange part B_{ab}^{ex}

$$B_{ab} = B_{ab}^{\mathrm{di}} + B_{ab}^{\mathrm{ex}}, \tag{7}$$

and to compute them separately. The exchange part of B_{ab} is convergent while the direct part is divergent. This is due to the long range behavior of the Coulomb interaction, which leads to collective effects. So let us first discuss the exchange part. This gives only a contribution for the electron-electron interaction, which, according to our earlier work [6], is in the weak field limit given by

$$B_{ee}^{\mathrm{ex}} = -\pi\lambda_{ee}^3\frac{\cosh(2x_e)}{\cosh^2(x_e)}E^0(\xi_{ee}) - \pi\lambda_{ee}^3\frac{x_e^2}{6}\frac{\cosh(2x_e)}{\cosh^2(x_e)}E^{\mathrm{B}}(\xi_{ee}). \tag{8}$$

We see that $x_e = \hbar\omega_{\mathrm{c}}^e/2kT$ plays the role of the small parameter in our theory. Furthermore, $E^0(\xi)$ is the virial function known before [3] and $E^{\mathrm{B}}(\xi)$ is a new magnetic exchange virial function first obtained in [6]:

$$E^{\mathrm{B}}(\xi_{aa}) = \sqrt{\pi}\sum_{k=0}^{\infty}\frac{1}{\Gamma(1 + k/2)}\left(\frac{k}{2+k}(1 - 2^{4-k})\zeta(k-3) - \frac{4}{2+k}(1 - 2^{2-k})\zeta(k-1)\right)\left(\frac{\xi_{aa}}{2}\right)^k. \tag{9}$$

The factor $\cosh(2x_e)/\cosh^2(x_e)$ in equation (9) resulted from the spins of the electrons. For higher temperatures we may cut the series after the second order that gives

$$E^{\mathrm{B}}(\xi) = \frac{\sqrt{\pi}}{8} - \frac{2}{3}\xi + \frac{\sqrt{\pi}}{2^6}\xi^2. \tag{10}$$

This is in agreement with the results given in an earlier work [5]. Let us now discuss the direct contribution. In the case of particles interacting *via* the Coulomb potential $V_{ab}(\mathbf{r}) = e_a e_b/(4\pi\epsilon_0|\mathbf{r}_a - \mathbf{r}_b|)$ the second virial coefficient defined by equations (6, 1) is divergent. In order to obtain a convergent expression one has to perform a screening procedure. Such a technique is well established in the zero magnetic field case [3,15] and can be extended to the nonzero magnetic field case. This program was also carried out by Cornu [11] and Boose and Perez [12], who used the Feynman-Kac formalism to derive a virial expansion for a magnetized multi-component system. Using the methods as described in [3,15], we have presented the second virial coefficient in [5,6] as a sum of contributions of lower orders in the interaction parameter e^{2n} with $n \leq 3$ and of high orders with $n > 3$. In this way, we have found an expression of the following form

$$B_{ab}^{\mathrm{di}} = B'_{ab} + B''_{ab} \tag{11}$$

where

$$B'_{ab} = -\frac{1}{4}\pi^{3/2}\lambda^3\xi_{ab}^2 h_2(x_a, x_b) - \frac{\pi}{3}\left(\frac{C}{2} + \ln 3 - \frac{1}{2}\right)\lambda^3\xi_{ab}^3 h_3(x_a, x_b). \tag{12}$$

The magnetic field correction h_2 satisfies $h_2 = 1$ if $B = 0$ and is at arbitrary magnetic field strengths explicitly given by:

$$h_2(x_a, x_b) = \left(\frac{1}{2} + \frac{4}{\pi}\int_0^1 \mathrm{d}t\,\sqrt{t(1-t)}\,(y_a + y_b) \times \frac{\operatorname{arctanh}\sqrt{1-(y_a+y_b)}}{\sqrt{1-(y_a+y_b)}}\right), \tag{13}$$

with $y_{a,b} = \lambda_{aa,bb}^2 \sinh(x_{a,b}t)\sinh(x_{a,b}(1-t))/(\lambda_{ab}^2 t(1-t)\,2x_{a,b}\sinh(x_{a,b}))$. In a good approximation (within accuracy of about 1 percent), we may use the formula

$$h_2(x_i, x_e) \simeq f_1(x_e) \tag{14}$$

with

$$f_1(x_e) = \frac{1}{2}\left[1 + \left(\frac{\tanh(bx)}{bx}\right)^c \frac{\operatorname{arctanh}\sqrt{1-\frac{\tanh(bx)}{bx}}}{\sqrt{1-\frac{\tanh(bx)}{bx}}}\right], \tag{15}$$

where the fitting parameters are found to be $c = 0.928\,52$ and $b = 0.404\,46$. In quadratic order in the field the function h_2 has the expansion

$$h_2(x_i, x_e) \approx h_2(0, x_e) \approx 1 - \frac{x_e^2}{48} + \ldots \tag{16}$$

The magnetic field correction h_3 is not explicitly known except in the limit of zero field, where $h_3 = 1$ holds. As it was shown in the zero field case, this term gives a small contribution to the second virial coefficient in the limit of low temperatures. Therefore, we can neglect this term in what follows. The higher order contributions to the second virial coefficient are given by

$$B''_{ab} = \frac{1}{2\Omega}P''\left(\frac{\Lambda_a^3}{2}\frac{\tanh(x_a)}{x_a}\right)\left(\frac{\Lambda_b^3}{2}\frac{\tanh(x_b)}{x_b}\right) \times \mathrm{Tr}\,(\mathrm{e}^{-\beta H_{ab}} - \mathrm{e}^{-\beta H_{ab}^0}). \tag{17}$$

Here, H_{ab} is the Hamilton operator of the two particle system and H_{ab}^0 of the noninteracting system. The additive term $\mu_{\mathrm{B}}^a B_0 \sigma_z$ describes the coupling between the intrinsic magnetic moment ($\mu_{\mathrm{B}}^a = e_a\hbar/(2m_a)$) of the charged particles and the magnetic field. The operator P'' means that all terms of order less than ξ^4 have to be omitted, since they have been taken into account in B'_{ab}. Formally the higher order contributions may be expressed by a resolvent expansion [3]

$$B''_{ab} = \frac{1}{2\Omega}\left(\frac{\Lambda_a^3}{2}\frac{\tanh(x_a)}{x_a}\right)\left(\frac{\Lambda_b^3}{2}\frac{\tanh(x_b)}{x_b}\right) \times \mathrm{Tr}\sum_k \frac{1}{2\pi}\int_C \mathrm{d}z\left[\frac{1}{H_{ab}^0 - z}V_{ab}\right]^k \frac{1}{H_{ab}^0 - z}\,. \tag{18}$$

The series may be written in the form

$$B''_{ab} = 2\pi^{3/2}\lambda_{ab}^3 \sum_{k=4}^{\infty}\frac{\zeta(k-2)h_k(x_a, x_b)}{\Gamma(1+k/2)}\left(\frac{\xi_{ab}}{2}\right)^k. \tag{19}$$

The functions h_k expressing the magnetic corrections satisfy the zero field condition

$$h_k(0, 0) = 1. \tag{20}$$

Therefore, in the zero magnetic field case, an exact calculation of the convergent second virial coefficient is possible in agreement with earlier works [3,4]. In the case of a weak magnetic field the second virial coefficient may be expanded with respect to x^2. Furthermore, we consider only the lowest order contribution and neglect higher order terms. The final result given in an earlier work [6] may be summarized as follows (with $a = i, e$):

$$B_{ae} = 2\pi\lambda_{ae}^3 Q^0(\xi_{ae}) + 2\pi\lambda_{ae}^3\frac{x_e^2}{24}Q^{\mathrm{B}}(\xi_{ae}). \tag{21}$$

Here, Q^0 is the well-known zero field virial function and Q^{B} is a new magnetic quantum virial function first derived in [6]

$$Q^{\mathrm{B}}(\xi) = \sqrt{\pi} \times \sum_{k=2}^{\infty}\frac{(k-3)\zeta(k-2) + (k-5)\zeta(k-4)}{\Gamma(2+k/2)}\left(\frac{\xi}{2}\right)^k. \tag{22}$$

Strictly speaking, the quantum virial function $Q^{\mathrm{B}}(\xi)$ has been proven so far only for the case of an infinite proton mass, *i.e.* one has to replace ξ_{ei} by $\xi_e = e^2/4\pi\epsilon_0 kT\lambda_e$ with $\lambda_e = \hbar/\sqrt{2m_e kT}$. This problem will however be of no importance for our calculations due to the small value of m_e/m_i.

$$\sigma^{\rm B}(\xi)=\sum_{n=1}^{\infty}2n^2\left(1+n^2\right)\left[{\rm e}^{\left(\frac{\xi}{2}\right)^2\frac{1}{n^2}}-1-\left(\frac{\xi}{2}\right)^2\frac{1}{n^2}-\frac{1}{2!}\left(\frac{\xi}{2}\right)^4\frac{1}{n^4}\right]$$
$$-\sum_{n=1}^{\infty}n^4\left(5+7n^2\right)\left(\frac{2}{\xi}\right)^2\left[{\rm e}^{\left(\frac{\xi}{2}\right)^2\frac{1}{n^2}}-1-\left(\frac{\xi}{2}\right)^2\frac{1}{n^2}-\frac{1}{2!}\left(\frac{\xi}{2}\right)^4\frac{1}{n^4}-\frac{1}{3!}\left(\frac{\xi}{2}\right)^6\frac{1}{n^6}\right] \quad (28)$$

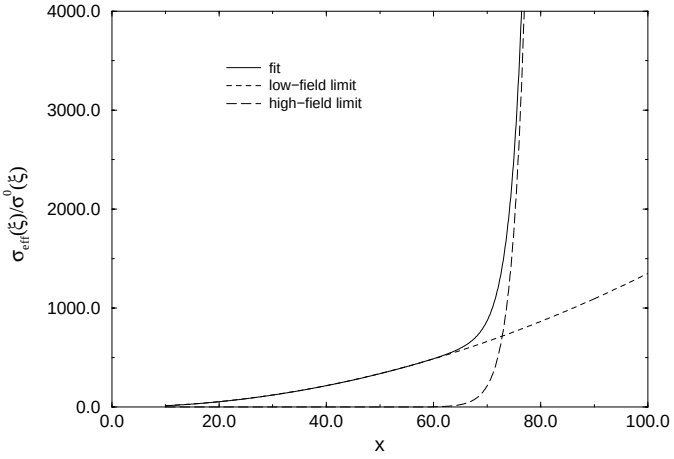

Fig. 1. Plot of the partition function, including fit formula, as a function of the magnetic field at a temperature of $T = 10^4$ K.

3 Ionization effects in the fugacity representation

Collecting the formulae equations (5, 21, 22) and completing this with the known relation between density and pressure

$$n_a=\tilde{z}_a\frac{{\rm d}(\beta p)}{{\rm d}\tilde{z}_a}, \quad (23)$$

we have an implicit closed relation for the pressure as a function of the density. This representation works at low as well as at high temperatures. Let us now consider the region, where bound states play a role. This means for the field-free case

$$(-E_{10}/k_{\rm B}T)=\xi_{ie}^2/4\gg 1, \quad (24)$$

where $E_{10}\simeq 13.6$ eV is the hydrogen ground state energy. In other words, we consider temperatures below 10^5 K. Restricting our study now to this region, we may simplify the representation given above by a procedure which is well established for the field free case [3,4] and which contains the steps:

(i) omitting the exchange terms,
(ii) omitting the terms of odd order in ξ,
(iii) keeping only asymptotic contributions to the virial functions [3,4],
(iv) excluding terms of the order $O(n^{5/2})$.

The fugacity expansion given above is well approximated by these assumptions and it transforms to the expression

$$\beta p=\beta p_{\rm id}+\frac{K^3}{12\pi}-\tilde{z}_i\tilde{z}_e\lambda_{ie}^3\left[\pi^{3/2}\xi_{ie}^2\left(1-\frac{x_e^2}{48}\right)-8\pi^{3/2}\sigma_{\rm eff}(T,B)\right]-... \quad (25)$$

Here, we have introduced the effective atomic partition function, which gives the asymptotic behavior of the second virial coefficient (22) at large values of ξ (see Ref. [6] for its derivation), and reads

$$\sigma_{\rm eff}(T,B)=\sigma^0(\xi_e)+\frac{x_e^2}{24}\left[\sigma^{\rm B}(\xi_e)+\frac{\xi_e^4}{192}\left(1+\frac{\pi^2}{3}\right)\right], \quad (26)$$

with the two contributions

$$\sigma^0(\xi)=\sum_{n=1}^{\infty}n^2\left[{\rm e}^{\left(\frac{\xi}{2}\right)^2\frac{1}{n^2}}-1-\left(\frac{\xi}{2}\right)^2\right], \quad (27)$$

and

see equation (28) above.

So far, this is just a definition. We essentially followed the idea developed by Brillouin-Planck-Larkin and others [3, 4] that the convergent interaction contributions of order higher than e^4 may be included into the atomic partition function.

The effective partition function given by equation (26) is valid only for small magnetic fields. On the other hand, asymptotic high-field expressions for the partition function are known [5]. Let us introduce the following simple interpolation formula for the effective partition function, which has the right behavior in both limits. Our approach is based on the knowledge of the weak-field and strong-field limit. We find for temperatures below 10^5 K

$$\sigma_{\rm eff}(T,B)=\sigma^0(\xi_e)+\frac{x_e^2}{24}\left[\sigma^{\rm B}(\xi_e)+\frac{\xi_e^4}{192}(1+\frac{\pi^2}{3})\right]+\exp\left(-\frac{1}{x}+\alpha\frac{\xi_e^2}{4}\ln^2(4x/\xi_e^2)\right). \quad (29)$$

The constant α has to be chosen in such a way, that the ground state energy of the hydrogen atom in a strong magnetic field is reproduced. Here, we have used the approximation $\alpha = 0.64$. The atomic partition function defined

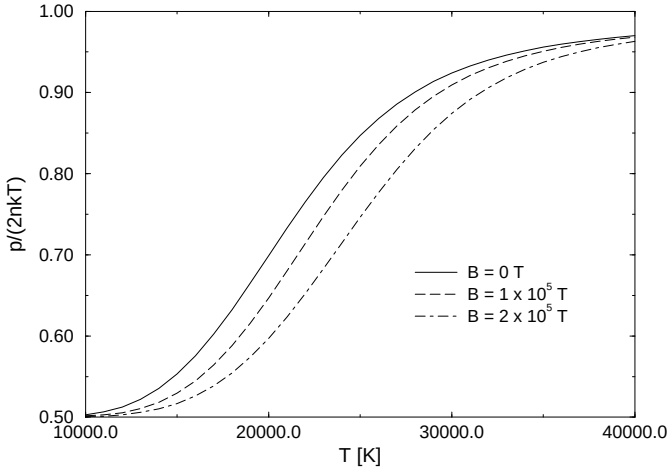

Fig. 2. The EOS calculated from the fugacity expansion as a function of temperature at a density of $n = 10^{25}$ m^{-3}.

in this way is graphically represented in Figure 1. For comparison we have also included the asymptotic expression

$$\sigma_{\rm eff}^{\rm asy}(T,B) = \exp(-E_0(B)/kT), \tag{30}$$

with $-E_0(B)/kT = (\alpha\xi_e^2/4)\ln^2(4x/\xi^2)$, and the weak-field results. We emphasis that our expressions (21, 22) is, strictly speaking, correct only up to the quadratic order in the magnetic field. Nevertheless, the interpolation formula equation (29) may be used for arbitrary field strength, and we may expect that it will give at least qualitatively reasonable results for any field strength. In the present (low-temperature) approximation we have symmetrical fugacities $\tilde{z}_e = \tilde{z}_i = \tilde{z}$ and get, finally, the following representation for the pressure

$$\beta p = 2\tilde{z} + \frac{\kappa^3}{12\pi} - \tilde{z}^2\pi^{3/2}\lambda_{ie}^2\xi_e^2\left(1 - \frac{x_e^2}{48}\right) + \tilde{z}^2 8\pi^{3/2}\lambda_{ie}^3\sigma_{\rm eff}(T,B) \tag{31}$$

and the density $n = n_e = n_i$

$$n = \tilde{z} + \frac{\kappa^3}{16\pi} - \tilde{z}^2\pi^{3/2}\lambda_{ie}^2\xi_e^2\left(1 - \frac{x_e^2}{48}\right) + \tilde{z}^2 8\pi^{3/2}\lambda_{ie}^3\sigma_{\rm eff}(T,B). \tag{32}$$

In this way, we have obtained closed expressions for the equation of state.

Figure 2 shows the pressure as a function of the temperature for a fixed density. With decreasing temperature the pressure decreases to half of its value. This reflects the formation of bound states. In spite of the fact that at the given temperature the effective partition function is extremely large, the contributions to the pressure remain finite due to the fact, that the fugacities become very small, simultaneously. In this way, a fugacity expansion can describe implicitly the formation of bound states and ionization effects in the same way as it was already known for the field-free case [1–3].

4 Transition to the chemical picture

In the case of a hydrogen plasma, the densities of electrons and ions are equal. In order to give a more explicit representation of the ionization equilibrium, we will derive a generalized Saha equation, which will be discussed in the next section. The general procedure of a transition to the chemical picture is well-known, and we will follow these lines [3,4]. The Saha equation is a special representation of the EOS for the region, where the binding energy is large in comparison with the thermal energy. The next step in the transition to chemical picture is a reinterpretation. We consider the bound states as composite particles, which must be treated on the same footing as elementary particles [3,16]. By inspection of the fugacity expansion (31), we reinterpret the term containing the effective partition function $\sigma_{\rm eff}(T,B)$ as the fugacity $z_0^\star$ of the neutral atoms

$$\tilde{z}_0^\star = \tilde{z}_i\tilde{z}_e 8\pi^{3/2}\lambda_{ie}^3\sigma_{\rm eff}(T,B). \tag{33}$$

Defining the fugacities of the free composite particles in the chemical picture by $\tilde{z}_e^\star = \tilde{z}_e$, $\tilde{z}_i^\star = \tilde{z}_i$, the pressure reads as follows

$$\beta p = \tilde{z}_e^\star + \tilde{z}_i^\star + \tilde{z}_0^\star + \frac{\kappa^{\star 3}}{12\pi} - \pi^{3/2}\tilde{z}_e^\star\tilde{z}_i^\star\lambda_{ei}^3\xi_{ei}^2(1 - \frac{x_e^2}{48}) + \tilde{z}_e^\star\tilde{z}_i^\star\,\frac{\pi}{2}\beta^3\frac{e_e^2e_i^4 + e_i^2e_e^4}{(4\pi\epsilon_0)^2}\,. \tag{34}$$

The particle densities of the new species are given by

$$n_e^\star = \tilde{z}_e^\star\frac{\partial(\beta p)}{\partial\tilde{z}_e^\star}, \quad n_i^\star = \tilde{z}_i^\star\frac{\partial(\beta p)}{\partial\tilde{z}_i^\star}, \quad n_0^\star = \tilde{z}_0^\star\frac{\partial(\beta p)}{\partial\tilde{z}_0^\star}\,. \tag{35}$$

Solving this equation by iteration, we find the chemical potentials

$$\begin{aligned}
\beta\mu_e &= \ln\left(n_e^\star\frac{\Lambda_e^3}{2}\frac{\tanh(x_e)}{x_e}\right) - \frac{1}{2}\frac{\beta e^2\kappa^\star}{4\pi\epsilon_0}(1 - \kappa^\star a)\,,\\
\beta\mu_i &= \ln\left(n_i^\star\frac{\Lambda_i^3}{2}\frac{\tanh(x_i)}{x_i}\right) - \frac{1}{2}\frac{\beta e^2\kappa^\star}{4\pi\epsilon_0}(1 - \kappa^\star a)\,,\\
\beta\mu_0 &= \ln\left(\frac{n_0^\star\Lambda^3}{4\sigma_{\rm eff}(T,B)}\right),
\end{aligned} \tag{36}$$

where now $\kappa^{\star 2} = (n_e^\star + n_i^\star)\beta e^2/\epsilon_0 = 2n_e^\star\beta e^2/\epsilon_0$.

We have further introduced the length parameter a, which may be interpreted as a kind of effective diameter of the charges. This parameter is defined by

$$a = \frac{4\pi\epsilon_0^2}{\beta^2e^4}\left[\pi^{\frac{3}{2}}\,\lambda_{ie}^3\,\xi_{ie}^2\,h_2(x_i,x_e)\right], \tag{37}$$

where the present low-field approximation yields

$$h_2(x_i,x_e) \approx (1 - \frac{x_e^2}{48}). \tag{38}$$

The parameter a is not only a function of the De Broglie wavelength, but also of the magnetic length $l_{\rm B}$. The magnetic field reduces the effective diameter. This reflects the

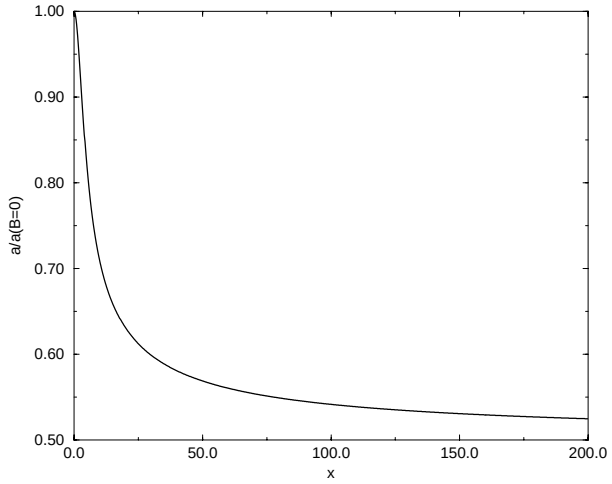

Fig. 3. Plot of the effective diameter as a function of the (dimensionless) magnetic field.

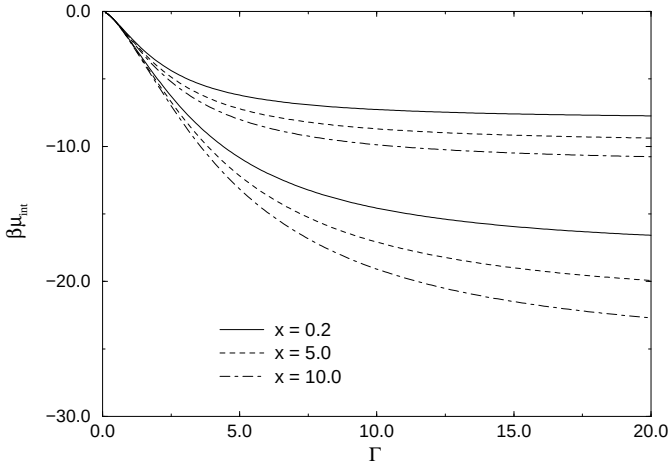

Fig. 4. The lowering of the ionization energy $\beta\mu_{\rm int}$ as a function of the density for different field strength (B[T] $\approx 1.49xT$[K]). The upper array of curves corresponds to $T =$ 50 000 K and the lower array of curves corresponds to $T =$ 10 000 K.

fact that the magnetic field localizes the particle perpendicular to field. The formulae equations (37, 38) are valid only for weak fields, *i.e.* for $x_e^2/48 \ll 1$. A heuristic approach to extend the formulae to higher fields may be based on the representation for the function $h_2(x_i, x_e)$ given earlier (see Eq. (13)). In Figure 3, the effective diameter of the electron a is represented as a function of the magnetic field. It is useful to extend the range of validity of equation (36) to larger values of $\kappa^* a$ by a kind of Padé approximation [3], which brings the chemical potential to the classical form assumed already in the twenties by Debye and Hückel [17]. We note that

$$\frac{\beta e^2\kappa^\star}{4\pi\epsilon_0}(1-\kappa^\star a) \approx \frac{\beta e^2\kappa^\star}{4\pi\epsilon_0}\frac{1}{(1+\kappa^\star a)}\,. \tag{39}$$

In this approximation, we find

$$\begin{aligned}\beta\mu_e &= \ln\left(n_e^\star\frac{\Lambda_e^3}{2}\frac{\tanh(x_e)}{x_e}\right) - \frac{1}{2}\frac{\beta e^2\kappa^\star}{4\pi\epsilon_0(1+\kappa^\star a)}\,,\\ \beta\mu_i &= \ln\left(n_i^\star\frac{\Lambda_i^3}{2}\frac{\tanh(x_i)}{x_i}\right) - \frac{1}{2}\frac{\beta e^2\kappa^\star}{4\pi\epsilon_0(1+\kappa^\star a)}\,.\end{aligned} \tag{40}$$

Combining these terms, we find for the interaction part of the plasma chemical potential

$$\beta\mu_{\rm int} = -\frac{\beta e^2\kappa^\star}{(4\pi\epsilon_0)(1+\kappa^\star a)}\,. \tag{41}$$

We will show in the next section that $\mu_{\rm int}$ plays the role of the lowering of the ionization energy and is therefore of central importance for the ionization phenomena in dense magnetized plasmas. From the Padé approximation for the chemical potential, we may calculate by standard thermodynamic relations the contributions to the free energy and the pressure. As already mentioned, the results take the Debye-Hückel form, which is known from the theory of electrolytes [17]. The interaction contributions to the free energy and to the pressure have in the present approximation the form

$$\begin{aligned}F_{\rm int} &= -kTV\frac{\kappa^{\star 3}}{12\pi}\tau(\kappa^\star a),\\ \beta p_{\rm int} &= -\frac{\kappa^{\star 3}}{24\pi}\phi(\kappa^\star a),\end{aligned} \tag{42}$$

with the functions $\tau(\kappa^\star a)$ and $\phi(\kappa^\star a)$ known from the Debye-Hückel theory [17,18]

$$\begin{aligned}\tau(x) &= \frac{3}{x^3}\left[\ln(1+x) - x + \frac{x^2}{2}\right],\\ \phi(x) &= \frac{3}{x^3}\left[-2\ln(1+x) + x + \frac{x}{1+x}\right].\end{aligned} \tag{43}$$

Instead of κ, we may use the dimensionless coupling strength $\Gamma = \beta e^2/(4\pi\epsilon_0 d)$ with $d = (3/4\pi n_i^\star)^{1/3}$. In terms of Γ, the chemical potential reads as

$$\beta\mu_{\rm int} = -6^{1/2}\frac{\Gamma^{3/2}}{1 + c\Gamma^{3/2}\tau^{1/2}f_1(x_e)} \tag{44}$$

with $\tau = 2\hbar^2 kT(4\pi\epsilon_0)^2/(m_{ie}e^4)$ and $c = \sqrt{6\pi}/8$.

In Figure 4, we have shown the chemical interaction potential $\beta\mu_{\rm int}$ as a function of the parameter Γ (calculated from the density of the density of the free ions) for different values of the temperature and the magnetic field.

We note that equations (41, 44) may violate the condition of thermodynamic stability [3] in certain regions of the temperature density plane

$$\frac{\partial\mu}{\partial n^\star} = \frac{2}{n^\star} + \frac{\partial\mu_{\rm int}}{\partial n^\star} > 0. \tag{45}$$

Let us study here this stability condition for the approximation (41). For this simple Debye-Hückel type formula,

the system allows a full analytical stability analysis [3, 17,18]. We find that the system is unstable in the region between the two branches of

$$\tilde{\mu}_{1,2} = \left(\frac{b^2}{8} - b\right) \pm \left[\left(\frac{b^2}{8} - b\right)^2 - b^2\right]^{1/2}. \quad (46)$$

Here, we have introduced the abbreviations

$$\tilde{\mu} = \left(\frac{8\pi n^\star \beta^3 e^6}{(4\pi\epsilon_0)^3}\right)^{1/2}, \quad b = \frac{\beta e^2}{4\pi\epsilon_0 a}\,. \quad (47)$$

The loss of stability with respect to density fluctuations is sometimes interpreted by means of a phase separation into a more dense and a less dense phase [3]. We note that the condition

$$b_{\rm cr} = \frac{\beta e^2}{4\pi\epsilon_0 T_{\rm cr}^{\rm B} a} = 16 \quad (48)$$

defines a critical temperature

$$T_{\rm cr}^{\rm B} = \frac{e^2}{64\pi\epsilon_0 kTa}, \quad (49)$$

which depends through a on the magnetic field. The parameter a is a monotonically decreasing function of the magnetic field strength (see Fig. 3). Therefore, the magnetic field increases the critical temperature. In other words, in the region of partial ionization the magnetic field tends to destabilize the plasma, *i.e.* to favor a separation into two phases. Typically $T_{\rm c}$ is in the region 15 000−30 000 K. For $T < T_{\rm c}^{\rm B}$ and strong coupling $\Gamma > 2$ the two-component plasma becomes unstable with respect to density fluctuations and may form two phases. According to the mass action law, both phases have different degrees of ionization. This reminds one of the phenomena known from electrolyte theory and from electron-hole plasmas [17,18].

We must emphasis that the stability analysis given above yields only a qualitative analysis of the phenomena of phase separation in dense plasmas. A more quantitative theory requires, in particular, a careful treatment of the charge-neutral interactions [19–22].

5 Saha equation and ionization equilibrium

Using the Debye-Hückel type approximation equation (41), we find from the condition of chemical equilibrium

$$\mu_e + \mu_i = \mu_0 \quad (50)$$

the modified Saha equation

$$\frac{n_0^\star}{n_e^\star n_i^\star} = \sigma_{\rm eff}(T,B)\,\Lambda_e^3 \exp\left(-\frac{\beta e^2 \kappa^\star}{4\pi\epsilon_0(1+\kappa^\star a)}\right). \quad (51)$$

Equation (51) differs from the ideal Saha equation [7] by an additional exponential factor depending on the chemical interaction potential. This quantity may be interpreted as the lowering of the ionization energy. The principal behavior of this important quantity is shown in Figure 4. The lowering of the ionization energy increases with the coupling strength (*i.e.* with the density) and increases with the magnetic field. At very large Γ, which is the classical limit, the dependence on the magnetic field is weakened. On the other hand, the ionization energy itself increases with the magnetic field. Both effects point in different directions. In general, the dependence of the ionization energy on the field dominates the behavior.

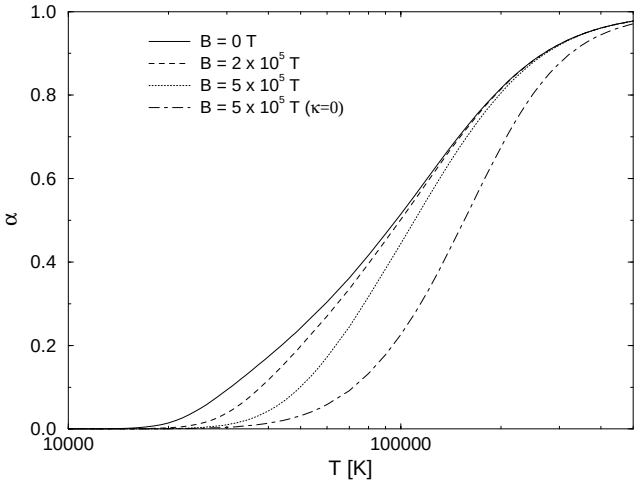

Fig. 5. The degree of ionization in dependence on the magnetic field at a density of $n = 10^{29}$ m^{-3}.

In Figure 5, the degree of ionization $\alpha = n_e^\star/n$ for a dense hydrogen plasma at various magnetic field strengths is plotted and compared with the results for the case of zero field and the degree of ionization derived from the ideal Saha equation [7]. We find an increase of the ionization degree in comparison with the ideal Saha equation [7] due to the nonideality effects. For densities of about 10^{29}–10^{30} m^3 the deviation from the ideal Saha equation may be as large as 10–15 % (see Fig. 5). At even higher densities, *i.e.*, $n \gg 10^{30}$ m^{-3}, the Saha equation given above may only be used as a rough approximation. The plasma can no longer be regarded as a weakly coupled system rather it must be treated as a strongly coupled system [3,20,22].

Additionally, we may characterize the dependence of the ionization degree on the magnetic field strength. With increasing magnetic field strength the degree of ionization decreases.

6 Conclusion

In this paper, we analyzed the ionization equilibrium in a hydrogen plasma in a weak constant uniform magnetic field. Starting from a fugacity expansion, we derived a general expression for the second virial coefficient as a series expansion with respect to the interaction parameter e^2 and the field B. Extending earlier results for the zero field case [3,4], we explicitly calculated the asymptotic

contributions for the case of strong interaction parameters $\xi \gg 1$. The results were used to establish the equation of state in the grand canonical ensemble. Finally, we have derived a generalized Saha equation, and we have shown that at higher densities nonideality effects can significantly increase the degree of ionization.

The accuracy of the absolute values of the EOS and the degrees of ionization considered here can be improved by calculating higher order contributions in the magnetic field to the virial coefficient. Nevertheless, the influence of the nonideality effects on the ionization equilibrium as shown in this paper remains approximately the same.

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References

1. G.P. Bartsch, W. Ebeling, Contr. Plasma Phys. **11**, 393 (1971).
2. F.J. Rogers, Phys. Rev. A **10**, 2441 (1974).
3. W. Ebeling, W.D. Kraeft, D. Kremp, *Theory of bound states and ionization equilibrium in plasmas and solids*, (Akademie-Verlag, Berlin, 1976).
4. W. Ebeling, Physica **38**, 378 (1968); *ibid.* **43**, 293 (1969).
5. M. Steinberg, J. Ortner, W. Ebeling, Phys. Rev. E **58**, 3806 (1998).
6. M. Steinberg, W. Ebeling, J. Ortner, Phys. Rev. E **61**, 2290 (2000).
7. D.L. Lai, E.E. Salpeter, Phys. Rev. A **52**, 2611 (1995).
8. V.K. Khersonskii, Sov. Astron. **31**, 225 (1987).
9. M.C. Miller, Mon. Not. R. Astron. Soc. **255**, 129 (1992).
10. A. Potekhin, G. Chabrier, Y.A. Shibanov, Phys. Rev. E **60**, 2193 (1999).
11. F. Cornu, Europhys. Lett. **37**, 591 (1997).
12. D. Boose, A. Perez, Phys. Lett. A **234**, 113 (1997).
13. A.Y. Potekhin, J. Phys. B **27**, 1073 (1994).
14. Yu.P. Kravchenko, M.A. Liberman, B. Johansson, Phys. Rev. A **54**, 287 (1996).
15. A.A. Vedenov, A.I. Larkin, Zhur. Eksptl. i Teoret. Fiz. **36**, 1133 (1959).
16. W. Ebeling, Physica **73**, 573 (1974).
17. B.P. Lee, M.E. Fisher, Phys. Rev. Lett. **76**, 2906 (1996).
18. H. Lehmann, W. Ebeling, Phys. Rev. E **54**, 2451 (1996).
19. D. Saumon, G. Chabrier, Phys. Rev. A **44**, 5122 (1991); *ibid.* **46**, 2084 (1992).
20. A. Potekhin, Phys. Plasmas **3**, 4156 (1996).
21. G. Chabrier, A. Potekhin, Phys. Rev. E **58**, 4941 (1998).
22. D. Beule, W. Ebeling, A. Förster, H. Juranek, S. Nagel, R. Redmer, G. Röpke, Phys. Rev. B **59**, 14177 (1999).